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CAVITATIONAL FLOWS AND GLOBAL INJECTIVITY OF CONFORMAL MAPS

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ABSTRACT. This paper treats some new mathematical aspects of the two-dimensional cavitational problem of the flow of a perfect fluid past an obstacle. Natural regularity conditions of very general type are found to ensure the global injectivity of the complex-potential and the presence of at most one zero of its derivative on the boundary of the flow. This derivative is the complex velocity. Previous authors have hypothesized the properties obtained here. The same regularity conditions are then shown to be satisfied by the classical solutions found via Villat's integral equation. A simple counterexample in §4 shows that the global injectivity of a holomorphic map defined on an unbounded Jordan domain cannot be deduced solely from its injectivity on the boundary. This simple fact raises new questions on the relation between cavitational flows and Villat's integral equation, which are discussed in §3.

1. INTRODUCTION

The main purpose of this paper is to clarify a number of fundamental issues connected with the use of conformal maps in the classical problem of a two-dimensional symmetric cavitational flow of an inviscid, incompressible fluid past a symmetric obstacle.

In the extensive literature on cavitation (cf. Birkhoff and Zarantonello (1957), Gilbarg (1960), Gurevich (1961), Wu (1972), Friedman (1982)) it has been systematically assumed that the complex potential Ω of the velocity field is globally injective on the unknown domain $\mathscr{F}$ occupied by the fluid and that there should be no stagnation points on its closure, except for a single point on the obstacle (the so-called "separation point"). In Theorem 2.46 we prove that it is unnecessary to hypothesize these properties, because they are in fact consequences of natural mathematical conditions on the unknown boundary. These conditions support the well-known conformal mapping method developed by Levi-Civita (1907). In §3 we prove that conversely these assumptions on the boundary of the flow are satisfied by the solutions obtained by means of the classical existence theory for the Villat integral equation. Levi-Civita's method is based upon the idea that the region occupied by the moving fluid can be

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determined as the image of a globally injective conformal map f defined on a half unit disk $\mathscr{H}$. Such a map f is then shown to be determined by its restriction on the circular part of $\partial\mathscr{H}$, which in turn is determined by a real-valued function l. The function l is then shown to satisfy Villat's integral equation. There is apparently a belief in the literature that once a solution of Villat's equation is given and consequently an injective map f on $\partial\mathscr{H}$ is defined, then f should be globally injective and determine a cavitational flow. But in §4 we show by means of a counterexample that these arguments are quite subtle and that a careful proof of this fact is required. Finally, we show that the flow problem, the boundary value problem for the conformal mapping of Levi-Civita, and Villat's equation are equivalent, in our mathematically precise setting. Our proofs exploit neither implicit assumptions nor "the physically reasonable assumptions" used in the literature, other than those explicitly stated ab initio. Such a precise formulation is very important for dealing with problems of solid-fluid interaction, as in Lanza and Antman (1990). I have not seen condition (2.2c) imposed on the regularity of the streamlines. This condition is crucial in our proofs. We refrain from treating existence and uniqueness, because they have been extensively discussed in the literature (cf. Leray (1935–36), Lavrentiev (1938), Serrin (1952a,b)).

Notation. Ordinary derivatives are indicated by primes. We denote the closure of a set $\mathscr{E}$ by cl $\mathscr{E}$ and the set of elements belonging to set $\mathscr{A}$ and not belonging to set $\mathscr{B}$ by $\mathscr{A}\backslash\mathscr{B}$. We denote by $\mathscr{D}(z_0, \varepsilon)$ the open disk $\{z \in \mathbb{C} : |z-z_0| < \varepsilon\}$. The space of holomorphic functions defined in an open set $\mathscr{A}$ is denoted by $\mathrm{H}(\mathscr{A})$. We call a function $f \in \mathrm{H}(\mathscr{A})$ biholomorphic if it is invertible, if $f(\mathscr{A})$ is open, and if its inverse $f^{-1} \in \mathrm{H}(f(\mathscr{A}))$. The space of m-times continuously differentiable functions on an interval such as $[-L, L]$ with the usual norm $\|\cdot\|_m$ is denoted $C^m[-L, L]$. When the domain of the functions under consideration is evident from the context, we do not bother including it in our notation for the space. The subspace of C^m whose functions have mth derivatives that are Hölder continuous with exponent α is denoted $C^{m,\alpha}$. If u is in $C^{0,\alpha}$ on a region Ω, then its α-Hölder quotient is $|u|_\alpha \equiv \sup\{|u(x) - u(y)|/|x-y|^\alpha,\ x, y \in \Omega,\ x \neq y\}$.

2. Formulation of the flow problem. Global injectivity of the complex potential

We consider a cavitational flow of an irrotational, incompressible, inviscid fluid under no external forces. The region $\mathscr{F}$ occupied by the fluid is assumed to be bounded by the curves ζ, γ, $\bar{\gamma}$. The given curve ζ represents the obstacle. The unknown curve γ, and its complex-conjugate $\bar{\gamma}$ represent two free streamlines issuing from edges of the obstacle. The curve ζ is assumed to satisfy

$$\begin{aligned} &\zeta \in \mathscr{Z} \cap C^{1,\alpha}[-L, L] \quad \text{for some } \alpha \in (0, 1], \\ (2.1) \qquad &\mathscr{Z} \equiv \{\zeta \in C^1[-L, L] : \zeta(\pm L) = \pm i,\ \operatorname{Im} \zeta' > 0, \\ &\qquad \zeta(s) = \bar{\zeta}(-s)\ \forall s \in [-L, L]\}. \end{aligned}$$

We consider only flows for which γ satisfies

$$|\gamma'| = 1, \quad \gamma(0) = i = \zeta(L), \quad \gamma'(0) = \frac{\zeta'(L)}{|\zeta'(L)|}, \quad \gamma'(\sigma) \to 1, \tag{2.2a}$$
$$\gamma(\sigma) \to \infty \quad \text{as } \sigma \to \infty, \qquad \operatorname{Im}\gamma > 0,$$
$$\gamma(\sigma) \notin \zeta([-L, L]) \quad \text{for } \sigma > 0, \quad \gamma \text{ is injective.}$$

To prescribe the regularity of γ, we set

$$\hat{\tau}(\sigma) \equiv \int_0^\sigma \left| \frac{d}{d\eta} \left(\frac{1}{\gamma(\eta)} \right) \right| d\eta, \qquad \hat{\sigma} \equiv \hat{\tau}^{-1}; \tag{2.2b}$$

$\hat{\tau}(\cdot)$ represents the arc length parameter of the curve $1/\gamma(\cdot)$ from $1/\gamma(0)$ to $1/\gamma(\sigma)$. We require that there be a $\beta \in (0, 1]$ such that

$$\gamma \in C^1[0, \infty), \quad \hat{\tau}(\infty) < \infty, \quad 1/\gamma(\hat{\sigma}(\cdot)) \in C^{1,\beta}[0, \hat{\tau}(\infty)]. \tag{2.2c}$$

(Thus the curve $1/\gamma$ is "Dini-smooth" in the sense of Pommerenke (1975, p. 298).)

The flow domain $\mathscr{F}$ is assumed to be the open connected component of $\mathbb{C}\backslash\{\zeta([-L, L]) \cup \gamma([0, \infty)) \cup \bar{\gamma}([0, \infty))\}$ containing the segment $(-\infty, \zeta(0))$:

$$\mathscr{F} \supseteq (-\infty, \zeta(0)). \tag{2.3}$$

The remaining connected component $\mathbb{C}\backslash \mathrm{cl}\mathscr{F}$ is assumed to be filled with a gas at constant pressure P.

It is well known (cf. Serrin (1959)) that the equations for the velocity $u + iv$ and pressure p governing the steady, irrotational, two-dimensional flow of an incompressible inviscid fluid with constant density ρ in the simply-connected domain $\mathscr{F}$ are equivalent to the existence of a complex potential Ω such that

$$\Omega \in \mathrm{H}(\mathscr{F}). \tag{2.4}$$

Once an arbitrary $c \in \mathbb{R}$ and Ω as in (2.4) are given, the complex velocity $w \equiv u - iv$ is determined by the equality

$$w(z) = \Omega'(z), \qquad z \in \mathscr{F}, \tag{2.5}$$

and the pressure by Bernoulli's formula

$$p(z) + \frac{\rho}{2}|\Omega'(z)|^2 = c, \qquad z \in \mathscr{F}. \tag{2.6}$$

To make sense of the boundary conditions, we require that Ω and Ω' admit continuous extensions to cl $\mathscr{F}$ with

$$\Omega, \Omega' \in C^0(\mathrm{cl}\mathscr{F}). \tag{2.7}$$

Then (2.6) implies that p can also be extended to cl $\mathscr{F}$. That the fluid cannot pass through $\partial\mathscr{F}$ is ensured by

$$\operatorname{Im}\Omega = 0 \quad \text{on } \partial\mathscr{F}. \tag{2.8}$$

We require the pressure and the velocity at infinity to equal $P \in \mathbb{R}$ and $U > 0$:

$$\Omega'(z) \to U, \quad p(z) \to P \quad \text{as } z \to \infty. \tag{2.9a,b}$$

Once Ω is given, (2.6) and (2.9) imply that

$$p(z) = P + \tfrac{\varrho}{2}U^2\{1 - U^{-2}|\Omega'(z)|^2\} \quad \text{for } z \in \mathrm{cl}\mathcal{F}. \tag{2.10}$$

In order that there be no discontinuity of the pressure across γ, $\overline{\gamma}$, equation (2.10) requires that

$$|\Omega'(\gamma(\sigma))| = U, \qquad \sigma \in [0, \infty). \tag{2.11}$$

We normalize the complex potential Ω (which is defined up to an arbitrary real additive constant) by

$$\Omega(\zeta(0)) = 0. \tag{2.12}$$

Our *flow problem* FP is to find the domain $\mathcal{F}$ and the function Ω such that (2.1)–(2.4), (2.7)–(2.9a), (2.11)–(2.12) are satisfied. We indicate a solution by $(\mathcal{F}, \Omega)$.

We emphasize that we make no assumptions either on the global injectivity of Ω, or on the zeros of Ω', or on the symmetry of Ω, as it is normally done in the literature. We shall show that these properties are implied by our assumptions. To do so, we need the following technical results.

2.13. **Lemma.** *Let* $\alpha \in (0, 1]$, *and* $\mathbb{N} \ni n \geq 2$. *If* $f \in C^0[a, b] \cap C^1(a, b]$ *and if* $\sup_{t\in(a,b]}|f'(t)t^{1-\alpha}|$ *is finite, then* $f \in C^{0,\alpha}[a, b]$. *If* $f \in C^1[a, b]$, *if* f *maps* $[a, b]$ *onto* $[c, d]$, *if* $f(a) = c$, *if* $f'(t) \neq 0$ *for* $t \in (a, b]$, *and if* $\sup_{t\in(a,b]}|f(t) - c|^{\frac{n-1}{n}}|f'(t)|^{-1}$ *is finite, then the inverse function* $f^{-1} \in C^{0,\frac{1}{n}}[c, d]$. *The corresponding statement holds if the roles of* a *and* b *are interchanged.*

Proof. Let $a < t_1 < t_2 \leq b$ and $c < y_1 < y_2 \leq d$. The first statement follows immediately from the inequality

$$|f(t_1) - f(t_2)| \leq \int_{t_1}^{t_2} t^{\alpha-1}\, dt \sup_{t\in(a,b]} |t^{1-\alpha}f'(t)|.$$

To prove the second observe that

$$\begin{aligned}|f^{-1}(y_1) - f^{-1}(y_2)| &\leq \int_{y_1}^{y_2} |\eta - c|^{\frac{1-n}{n}}\, d\eta \sup_{\eta\in(c,d]} \left[|\eta - c|^{\frac{n-1}{n}}|f'(f^{-1}(\eta))|^{-1}\right] \\ &= \int_{y_1}^{y_2} |\eta - c|^{\frac{1-n}{n}}\, d\eta \sup_{t\in(a,b]} \left[|f(t) - c|^{\frac{n-1}{n}}|f'(t)|^{-1}\right]. \quad \square\end{aligned}$$

Let

$$\mathcal{I}^+ \equiv \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}, \qquad \mathcal{I}^- \equiv \{z \in \mathbb{C} : \mathrm{Im}\, z < 0\}. \tag{2.14}$$

2.15. **Lemma.** *Let* $U \geq 0$ *and let FP have a solution* $(\mathcal{F}, \Omega)$. *Then*

$$\lim_{\sigma\to\infty} \frac{\gamma(\sigma)}{|\gamma(\sigma)|} = 1, \qquad \lim_{\tau\to\hat{\tau}(\infty)} \frac{d}{d\tau}\left\{\frac{1}{\gamma(\hat{\sigma}(\tau))}\right\} = -1. \tag{2.16a,b}$$

Moreover, there exists a unique homeomorphism

$$F : \mathrm{cl}\mathscr{J}^+ \to \mathrm{cl}\mathscr{F} \cap \mathrm{cl}\mathscr{J}^+ , \tag{2.17}$$

depending only on $\mathscr{F}$ *(and independent of* U, Ω), *such that*

$$F : \mathscr{J}^+ \to \mathscr{F} \cap \mathscr{J}^+ \quad \textit{is biholomorphic}, \tag{2.18}$$

$$F(0) = \zeta(0), \quad F(1) = i, \quad \lim_{z\to\infty} F(z) = \infty. \tag{2.19a,b,c}$$

The holomorphic map F' *can be extended to* $\mathrm{cl}\mathscr{J}^+\backslash\{0\}$ *and satisfies*

$$\begin{gathered} F'(z) \neq 0 \quad \textit{if } z \in \mathrm{cl}\mathscr{J}^+\backslash\{0\}, \\ F'(\infty) \equiv \lim_{z\to\infty} F'(z) \in (0, \infty), \qquad \lim_{z\to\zeta(0)} (F^{-1}(z))' = 0. \end{gathered} \tag{2.20}$$

Proof. We first prove (2.16a). Since $\lim_{\sigma\to\infty} \gamma' = 1$, for every $\varepsilon > 0$ there exists a $\sigma_1 > 0$ such that $\sigma \geq \sigma_1$ implies that $\mathrm{Re}\,\gamma' > 1 - \varepsilon$, $|\mathrm{Im}\,\gamma'| < \varepsilon$, and $\mathrm{Re}\,\gamma(\sigma) > 0$. Hence

$$0 \leq \limsup_{\sigma\to\infty} \left|\frac{\mathrm{Im}\,\gamma(\sigma)}{\mathrm{Re}\,\gamma(\sigma)}\right| \leq \limsup_{\sigma\to\infty} \frac{|\mathrm{Im}\,\gamma(\sigma_1)| + (\sigma - \sigma_1)\varepsilon}{(1-\varepsilon)(\sigma - \sigma_1) - \mathrm{Re}\,\gamma(\sigma_1)} = \frac{\varepsilon}{1-\varepsilon}. \tag{2.21}$$

The arbitrariness of ε implies that $\lim_{\sigma\to\infty} \mathrm{Im}\,\gamma(\sigma)/\mathrm{Re}\,\gamma(\sigma) = 0$. Hence (2.16a) follows. We also have

$$\lim_{\tau\to\hat{\tau}(\infty)} \frac{d}{d\tau}\left\{\frac{1}{\gamma(\hat{\sigma}(\tau))}\right\} = -\lim_{\tau\to\hat{\tau}(\infty)} \frac{\gamma'(\hat{\sigma}(\tau))}{\gamma^2(\hat{\sigma}(\tau))} \frac{|\gamma^2(\hat{\sigma}(\tau))|}{|\gamma'(\hat{\sigma}(\tau))|} = -1, \tag{2.22}$$

by (2.16a) and (2.2a). Let $x \in \mathbb{R}$ and $x > |\zeta(0)|$. It is easy to see that the fractional transformation

$$R(z) \equiv \frac{(zx+1)}{(z-x)} \tag{2.23}$$

maps $\mathrm{cl}\{\mathscr{J}^+ \cap \mathscr{F}\}$ homeomorphically onto a subset of $\mathbb{C}$ bounded by a simple closed curve. The membership of $\zeta \in C^{1,\alpha} \cap \mathscr{Z}$ and conditions (2.2), (2.3), (2.16b) ensure that $R(\partial\{\mathscr{F} \cap \mathscr{J}^+\}) \cup \{x\}$ can be parametrized by a simple closed curve $\Gamma \in C^{1,\min\{\alpha,\beta\}}[0, \tau_3]$, $\tau_3 > 0$, such that

$$|\Gamma'| = 1, \qquad \Gamma([0, \tau_3]) = R(\partial\{\mathscr{F} \cap \mathscr{J}^+\}) \cup \{x\}, \tag{2.24}$$

$$\begin{gathered} \Gamma(0) = \Gamma(\tau_3) = R(\zeta(0)), \qquad \Gamma'(0) = -i, \\ \Gamma'(\tau_3) = -1, \qquad \Gamma(\tau_1) = R(i) = -i, \\ \Gamma(\tau_2) = x, \quad \text{for some } 0 < \tau_1 < \tau_2 < \tau_3. \end{gathered} \tag{2.25}$$

We denote by $\mathscr{I}[\Gamma]$ the bounded connected component of $\mathbb{C}\backslash\Gamma([0, \tau_3])$ enclosed by Γ. Accordingly, R maps $\mathrm{cl}\{\mathscr{J}^+ \cap \mathscr{F}\}$ onto $\mathrm{cl}\,\mathscr{I}[\Gamma]$. As a next step to define F, we now "flatten" the corner of Γ at $\tau = 0$. To do so we consider the map

$$G(z) = [z - R(\zeta(0))]^2. \tag{2.26}$$

Since

$$\text{cl}\,\mathscr{I}[\Gamma] \subseteq [R(\zeta(0)), \infty) \cup \mathscr{I}^{-}, \tag{2.27}$$

the map G is a homeomorphism of cl $\mathscr{I}[\Gamma]$ onto the closure of the set $\mathscr{I}[\Gamma_1]$ bounded by the curve

$$\Gamma_1(\tau) \equiv (\Gamma(\tau) - R(\zeta(0)))^2, \qquad \tau \in [0, \tau_3]. \tag{2.28}$$

Γ_1 is clearly a simple, closed curve of class $C^{1,\min\{\alpha,\beta\}}$. Now, let

$$\tilde{\mu}(\tau) \equiv \int_0^\tau |\Gamma_1'(\tau)|\,d\tau, \qquad \tau \in [0, \tau_3], \quad \tilde{\tau} \equiv (\tilde{\mu})^{-1}. \tag{2.29}$$

We want to prove that

$$\Gamma_2 \equiv \Gamma_1 \circ \tilde{\tau} \in C^{1,\min\{\alpha,\beta\}/2}[0, \tilde{\mu}(\tau_3)] \tag{2.30}$$

and that

$$\Gamma_2(0) = \Gamma_2(\tilde{\mu}(\tau_3)), \qquad \frac{d}{d\mu}\Gamma_2(0) = \frac{d}{d\mu}\Gamma_2(\tilde{\mu}(\tau_3)). \tag{2.31a,b}$$

Conditions (2.30) and (2.31) ensure that the curve Γ_2 is Dini-smooth. Since Γ_1 is a closed curve, equation (2.31a) holds. To prove (2.30), (2.31b), we observe that

$$\frac{d}{d\mu}\Gamma_2(\mu) = \frac{\Gamma(\tilde{\tau}(\mu)) - R(\zeta(0))}{|\Gamma(\tilde{\tau}(\mu)) - R(\zeta(0))|}\Gamma'(\tilde{\tau}(\mu)), \qquad \mu \in (0, \tilde{\mu}(\tau_3)). \tag{2.32}$$

Then (2.31b) follows immediately. Since (2.32) holds, to prove (2.30) it suffices to show that

$$\tilde{\tau} \in C^{0,1/2}[0, \tilde{\mu}(\tau_3)], \qquad \frac{a+ib}{|a^2+b^2|^{1/2}} \in C^{1,\min\{\alpha,\beta\}}[0, \tau_3], \tag{2.33a,b}$$

where $a(\tau) + ib(\tau) \equiv \Gamma(\tau) - R(\zeta(0))$. Property (2.33a) follows from Lemma 2.13 applied to the intervals $[0, \tau_3/2]$, $[\tau_3/2, \tau_3]$. We now consider (2.33b). We first show that $a|a^2+b^2|^{-1/2} \in C^{1,\min\{\alpha,\beta\}}[0, \tau_3/2]$. Clearly

$$\frac{d}{d\tau}\left\{\frac{a}{|a^2+b^2|^{1/2}}\right\} = \frac{b(a'b - ab')}{|a^2+b^2|^{3/2}} \in C^0(0, \tau_3/2]. \tag{2.34}$$

By Lemma 2.13, it suffices to show that

$$\lim_{\tau\to 0^+} \tau^{1-\min\{\alpha,\beta\}}\frac{d}{d\tau}\left\{\frac{a}{|a^2+b^2|^{1/2}}\right\} \in \mathbb{R}. \tag{2.35}$$

Since $\lim_{\tau\to 0^+} |(a+ib)\tau^{-1}| = 1$ and $\lim_{\tau\to 0^+} b\tau^{-1} = b'(0)$, the limit (2.35) follows from

$$\sup_{\tau\in(0,\tau_3/2]} |\tau^{-\min\{\alpha,\beta\}-1}(a'b - ab')| \in \mathbb{R}. \tag{2.36}$$

Since $a, b \in C^{1,\min\{\alpha,\beta\}}$, and $a(0) + ib(0) = 0$, we can use the Mean Value Theorem to obtain

$$(2.37)\qquad \begin{aligned} |b(\tau) - b'(0)\tau| &= |b'(\tau') - b'(0)|\tau \le |b'|_{\min\{\alpha,\beta\}}\tau^{1+\min\{\alpha,\beta\}}, \\ |a(\tau) - a'(0)\tau| &= |a'(\tau'') - a'(0)|\tau \le |a'|_{\min\{\alpha,\beta\}}\tau^{1+\min\{\alpha,\beta\}}, \end{aligned}$$

for some $\tau', \tau'' \in (0, \tau)$. Hence (2.36) follows from the equality

$$(2.38)\qquad \begin{aligned} &b(\tau)a'(\tau) - a(\tau)b'(\tau) \\ &\quad = (b(\tau)a'(\tau) - b'(0)\tau a'(\tau)) + (b'(0)\tau a'(\tau) - b'(0)\tau a'(0)) \\ &\qquad + (b'(0)\tau a'(0) - b'(\tau)\tau a'(0)) + (b'(\tau)\tau a'(0) - b'(\tau)a(\tau)). \end{aligned}$$

We similarly show that $a|a^2 + b^2|^{-1/2} \in C^{1,\min\{\alpha,\beta\}}[0, \tau_3/2]$ and that $b|a^2 + b^2|^{-1/2} \in C^{1,\min\{\alpha,\beta\}}[0, \tau_3]$. Thus the proof of (2.30), (2.31) is complete. By invoking the Riemann Mapping Theorem, we deduce that for every $z_0 \in \mathscr{I}[\Gamma_2]$ (the bounded connected component of $\mathbb{C}\backslash\Gamma_2([0, \tau_3])$), there exists a unique biholomorphic map

$$(2.39)\qquad g : \mathscr{D}(-i/2, 1/2) \to \mathscr{I}[\Gamma_2],$$

such that

$$(2.40)\qquad g(-i/2) = z_0, \qquad g'(-i/2) > 0.$$

By a result of Carathéodory (cf. Pommerenke (1975, Theorem 9.10)), g can be extended to $\mathrm{cl}\,\mathscr{D}(-i/2, 1/2)$ and

$$(2.41)\qquad g : \mathrm{cl}\,\mathscr{D}(-i/2, 1/2) \to \mathrm{cl}\,\mathscr{I}[\Gamma_2] \quad \text{is a homeomorphism.}$$

By taking the composition of g with a suitable fractional transformation of $\mathrm{cl}\,\mathscr{D}(-i/2, 1/2)$ onto itself, we can certainly assume g to be normalized by

$$(2.42)\qquad g(0) = [x - R(\zeta(0))]^2, \quad g(-i) = 0, \quad g((1-i)/2) = [R(i) - R(\zeta(0))]^2,$$

instead of by (2.40). Since (2.30) and (2.31) hold, a well-known result of Warschawski (cf. Pommerenke (1975, Theorem 10.2)) yields that

$$(2.43)\qquad \begin{aligned} &g' \text{ can be extended with continuity to } \mathrm{cl}\,\mathscr{D}(-i/2, 1/2), \\ &\qquad g'(\omega) \ne 0 \quad \text{if } \omega \in \mathrm{cl}\,\mathscr{D}(-i/2, 1/2). \end{aligned}$$

Since $(z + i)^{-1}$ maps $\mathrm{cl}\,\mathscr{J}^+$ homeomorphically onto $\mathrm{cl}\,\mathscr{D}(-i/2, 1/2)\backslash\{0\}$, we can construct F as in the statement of the lemma. Namely, let

$$(2.44)\qquad F(z) \equiv R^{-1} \circ G^{-1} \circ g(1/(z+i)), \qquad z \in \mathrm{cl}\,\mathscr{J}^+.$$

A simple verification based on the properties of R, G, g shows that F satisfies (2.17)–(2.20). We just remark that if F_1 and F_2 both satisfy (2.17)–(2.20), then by setting

$$(2.45)\qquad \begin{aligned} &g_j(0) = [x - R(\zeta(0))]^2, \qquad g_j(w) \equiv G \circ R \circ F_j((1 - iw)/w), \\ &\qquad w \in \mathrm{cl}\,\mathscr{D}(-i/2, 1/2)\backslash\{0\}, \quad j = 1, 2, \end{aligned}$$

we obtain two biholomorphic functions g_1, g_2 satisfying (2.39), (2.41), (2.42). Hence, the Riemann Mapping Theorem implies that $g_1 = g_2$ and consequently that $F_1 = F_2$. □

We are now ready to prove

2.46. **Theorem.** *Let* $U \geq 0$ *and let FP have a solution* $(\mathscr{F}, \Omega)$. *Then*

$$\Omega(z) = UF'(\infty)F^{-1}(z), \qquad z \in \mathrm{cl}\mathscr{F} \cap \mathrm{cl}\mathscr{J}^+, \tag{2.47}$$

where F *depends solely on* $\mathscr{F}$ *and is defined as in Lemma* 2.15. *Furthermore,*

$$\Omega(\bar{z}) = \overline{\Omega(z)}, \qquad z \in \mathrm{cl}\mathscr{F}. \tag{2.48}$$

If $U > 0$, *then*

$$\Omega'(z) \neq 0 \quad \textit{for } z \in \mathrm{cl}\mathscr{F}\setminus\{\zeta(0)\}, \tag{2.49}$$

$$\Omega : \mathscr{F} \to \mathbb{C} \quad \textit{is injective.} \tag{2.50}$$

Proof. The function

$$\Upsilon(z) \equiv \frac{\Omega(F(z))}{F'(\infty)}, \qquad z \in \mathrm{cl}\mathscr{J}^+, \tag{2.51}$$

is holomorphic in $\mathscr{J}^+$, continuous on cl $\mathscr{J}^+$, and satisfies Im $\Upsilon = 0$ if $z \in \mathbb{R}$. By the Schwarz Reflection Principle, Υ can be extended to a holomorphic function on $\mathbb{C}$, which we still denote by Υ. Clearly $\lim_{z\to\infty} \Upsilon'(z) = U$. Hence $\Upsilon'(z) = U$ $\forall z \in \mathbb{C}$ by Liouville's Theorem. Conditions (2.12) and (2.19a) imply that $\Upsilon(0) = 0$. Consequently $\Upsilon(z) = Uz$ and (2.47) follows. Now let $(\mathscr{F}, \Omega)$ be a solution of FP. Then $(\mathscr{F}, \overline{\Omega(\bar{z})})$ is also a solution of FP. By (2.47), we have that $\Omega(z) = \overline{\Omega(\bar{z})}$, $z \in \mathrm{cl}\mathscr{F} \cap \mathrm{cl}\mathscr{J}^+$. Then (2.48) follows by the Identity Principle. Finally, (2.49) and (2.50) are consequences of (2.47), (2.48) and the properities of F. □

If $U = 0$, then Theorem 2.46 implies that $\Omega = 0$. It is then clear that any domain $\mathscr{F}$ bounded by ζ, γ, $\bar{\gamma}$ as in (2.2) would be a solution for $U = 0$.

Remark. The proofs of Lemma 2.15 and Theorem 2.46 could be carried out if hypothesis (2.2c) were to be replaced with the more general requirement that the curve $1/\gamma(\hat{\sigma})$ be Dini-smooth. Indeed Dini-smoothness suffices to apply Warschawski's Theorem as done above (cf. Pommerenke (1975, p. 298 and Theorem 10.2)). To avoid unnecessary complications in the proof of Lemma (2.15), we did not carry out this minor generalization. We also point out that if $U > 0$, then the condition $\lim_{\sigma\to\infty} \gamma'(\sigma) = 1$ can be shown to follow from the other assumptions of FP together with the weaker condition that $\lim_{\sigma\to\infty} \gamma'(\sigma) \neq -1$. For the sake of brevity, we omit a proof of this simple fact.

3. The flow problem, the conformal mapping of Levi-Civita, and Villat's equation

We now reduce FP to a boundary-value problem for a conformal mapping f from

$$\mathscr{H} \equiv \{\omega \in \mathbb{C} : |\omega| < 1, \mathrm{Re}\,\omega > 0\} \tag{3.1}$$

onto $\mathscr{F}$. By using the generalized Schwarz-Christoffel transformation of Gilbarg (1949), we can verify that the function

$$h(\omega) \equiv (2 - \omega^2 - \omega^{-2})/4, \qquad h : \mathrm{cl}\,\mathscr{H}\backslash\{0\} \to \mathbb{C}, \tag{3.2}$$

is the unique biholomorphic map on $\mathscr{H}$ such that

$$\begin{gathered} h(1) = 0, \quad h(i) = 1, \quad \lim_{\omega\to 0} h(\omega) = \infty, \\ h(\omega) = \overline{h(\overline{\omega})}, \qquad h(\mathscr{H}) = \mathbb{C}\backslash[0, \infty). \end{gathered} \tag{3.3}$$

Then we have

3.4. **Proposition.** *Let* $U > 0$, $\alpha \in (0, 1]$, *and* $\zeta \in C^{1,\alpha} \cap \mathscr{Z}$. *If FP has a solution* $(\mathscr{F}, \Omega)$, *then there is a unique map* f *corresponding to it such that*

$$f : \mathscr{H} \to \mathbb{C} \quad \textit{is holomorphic}, \tag{3.5a}$$

$$f : \mathrm{cl}\,\mathscr{H}\backslash\{0\} \to \mathrm{cl}\, f(\mathscr{H}) \quad \textit{is a homeomorphism}, \tag{3.5b}$$

$$f(\mathscr{H}) = \mathscr{F}, \tag{3.5c}$$

$$\begin{gathered} f' \textit{ has a continuous extension to } \mathrm{cl}\,\mathscr{H}\backslash\{0\}, \\ \textit{satisfying } f'(\omega) \neq 0 \textit{ for } \omega \in \mathrm{cl}\,\mathscr{H}\backslash\{0, \pm i\}, \end{gathered} \tag{3.5d}$$

$$\lim_{\omega\to\pm i} \left\{ \frac{f'(\omega)}{\omega \mp i} \right\} \in \mathbb{C}\backslash\{0\}, \tag{3.5e}$$

(3.5f) *there exists a* $\beta \in (0, 1]$ *such that* $\tau^\sharp(0) < \infty$, $1/f(-it^\sharp(\cdot)) \in C^{1,\beta}[0, \tau^\sharp(0)]$, *where* $\tau^\sharp(t) \equiv \int_{-1}^{t} \left| \frac{d}{d\eta}\left(\frac{1}{f(-i\eta)}\right)\right| d\eta$, $t^\sharp \equiv (\tau^\sharp)^{-1}$,

$$f(\pm i) = \pm i, \qquad \lim_{\omega\to 0} f(\omega) = \infty, \tag{3.5g}$$

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \ni t \mapsto f(e^{it}) \in \zeta([-L, L]) \quad \textit{is surjective}, \tag{3.5h}$$

$$f(\omega) = \overline{f(\overline{\omega})}, \qquad \omega \in \mathscr{H}, \tag{3.5i}$$

$$\begin{gathered} M \equiv \lim_{\omega\to 0} U\omega^3 f'(\omega) \quad \textit{is real and positive}, \\ |f'(-it)| = \frac{M}{U}\frac{t^4 - 1}{t^3} \quad \forall\, t \in [-1, 0). \end{gathered} \tag{3.5j}$$

Conversely, if f *is a given function satisfying* (3.5), *except for* (3.5c), *then a solution of FP is defined by*

$$\mathscr{F}[\zeta, U] \equiv f(\mathscr{H}), \tag{3.6a}$$

$$\Omega[\zeta, U](z) \equiv \frac{M}{2}\left\{2 - [f^{-1}(z)]^2 - [f^{-1}(z)]^{-2}\right\}, \qquad z \in \mathrm{cl}\, f(\mathscr{H}). \tag{3.6b}$$

(By the uniqueness of f asserted in the second sentence of this proposition, the f generated by (3.6) *is the same f as that appearing in* (3.6).) *Furthermore, if FP were to have two distinct solutions for the same data ζ, U, then the corresponding f's would be distinct.*

Proof. Let FP have a solution $(\mathscr{F}, \Omega)$ and let the function F be defined as in Lemma 2.15. The map

$$f(\omega) \equiv F(h(\omega)), \qquad \omega \in \mathrm{cl}\,\mathscr{J}^+ \cap (\mathrm{cl}\,\mathscr{H} \backslash \{0\}), \tag{3.7}$$

is real on $\mathbb{R} \cap (\mathrm{cl}\,\mathscr{H} \backslash \{0\})$ and can therefore be extended by reflection to all of $\mathrm{cl}\,\mathscr{H} \backslash \{0\}$. Properties (3.5a–j) follow directly from the definitions of F, h, f (cf. Lemma 2.15, (3.2), (3.7)). From (3.7) we have

$$M \equiv \lim_{\omega \to 0} U\omega^3 \frac{d}{d\omega}\{F(h(\omega))\} = \frac{U}{2} F'(\infty). \tag{3.8}$$

Hence, by (2.47), (3.2), and (3.7), we deduce (3.6b). By differentiating (3.6b) with respect to z, evaluating the resulting expression at $z = f(-it)$, and by applying (2.11), we obtain (3.5k).

We now assume that f is given as in (3.5) except for (3.5c). We first verify the conditions on $\mathscr{F} \equiv f(\mathscr{H})$. Let

$$\sigma^*(t) \equiv \int_{-1}^{t} |f'(-i\xi)|\, d\xi, \qquad t^* \equiv (\sigma^*)^{-1}, \tag{3.9}$$

and

$$\gamma(\sigma) \equiv f(-it^*(\sigma)), \qquad \sigma \in [0, \infty). \tag{3.10}$$

Clearly

$$f(\partial\mathscr{H} \backslash \{0\}) = \zeta([-L, L]) \cup \gamma([0, \infty)) \cup \overline{\gamma}([0, \infty)). \tag{3.11}$$

The injectivity of f and condition (3.5i) imply that $\mathrm{Im}\,\gamma > 0$. We deduce that $\lim_{\sigma \to \infty} \gamma'(\sigma) = 1$ from (3.9), (3.10), and (3.5j). To verify that $\gamma'(0) = \zeta'(L)/|\zeta'(L)|$ we first equate the limit in (3.5e) computed with $\omega = -it$, $t \to -1^+$, to the same limit computed with $\omega = e^{it}$, $t \to \frac{\pi}{2}^-$. Then we use (3.9), (3.10), and the parametrization of ζ given by $t \mapsto f(e^{it})$. It is straightforward to verify the remaining conditions listed in (2.2). We now prove that

$$\partial\{f(\mathscr{H})\} \subseteq f(\partial\mathscr{H} \backslash \{0\}). \tag{3.12}$$

Let $z \in \partial f(\mathscr{H})$. Then $z = \lim_{n \to \infty} f(\omega_n)$, $\omega_n \in \mathscr{H}$. Since $\mathrm{cl}\,\mathscr{H}$ is compact, we can assume that $\omega_n \to \omega \in \mathrm{cl}\,\mathscr{H}$ by selecting a suitable subsequence. By (3.5g), $\omega \neq 0$. Since $f'(\omega) \neq 0$ in $\mathscr{H}$, $f(\mathscr{H})$ is open. Hence $\omega \in \partial\mathscr{H} \backslash \{0\}$. Now, let A_1 and A_2 be the connected component of $\mathbb{C} \backslash f(\partial\mathscr{H} \backslash \{0\})$ containing $(-\infty, \zeta(0))$ and $(\zeta(0), \infty)$, respectively. Condition (3.12) and the connectedness of $f(\mathscr{H})$ imply that only one of the following alternatives is possible

$$f(\mathscr{H}) = A_1, \quad f(\mathscr{H}) = A_2, \quad f(\mathscr{H}) = \mathbb{C}. \tag{3.13a,b,c}$$

By (3.5i) and the injectivity of f, we have that $\operatorname{cl} f(\mathscr{H}) \cap \mathbb{R} = f((0, 1])$. Then the injectivity of f and conditions (3.5g, h, i) imply that

$$\text{(3.14a,b)} \qquad \text{either} \quad f((0, 1]) = (-\infty, \zeta(0)] \quad \text{or} \quad f((0, 1]) = [\zeta(0), \infty)$$

and that consequently only (3.13a, b) can hold. Let $c \in (0, 1]$. Since $f(\pm i) = \pm i$ and since the holomorphic and injective function f preserves orientation on the simple curve composed of the arc $\partial\mathscr{H}$, the segment $[c, i]$, and the segment $[c, -i]$, we must have $f(c) < \zeta(0)$. Hence (3.14a) holds and consequently $f(\mathscr{H}) = A_1$. Thus we have proved that $f(\mathscr{H})$ is the domain bounded by $\zeta, \gamma, \bar{\gamma}$ and that (2.3) holds. By using (3.5a, b, j, k),we can easily verify (2.4), (2.7)–(2.9a), (2.11), (2.12). Now let $(\mathscr{F}_j, \Omega_j)$, $j = 1, 2$, be two distinct solutions of FP. Since equality of the $\mathscr{F}_j$'s would imply that $f_1 = f_2$ and hence that $\Omega_1 = \Omega_2$, we must have $\mathscr{F}_1 \neq \mathscr{F}_2$, which in turn implies that $f_1 \neq f_2$. Finally, it is clear that the solution given in (3.6) also satisfies (3.5c) and so all of (3.5) by hypothesis. Then by the first part of the statement, f is the "f" corresponding to solution (3.6). □

Thus FP is reduced to the *conformal mapping problem* CMP: Find f satisfying (3.5) except for (3.5c).

For a given $\zeta \in \mathscr{Z}$, let the continuous function $\phi[\zeta](s)$ (which we abbreviate as $\phi(s)$) be the counterclockwise angle that $\zeta'(s)$ makes with the positive imaginary axis:

$$\text{(3.15)} \qquad \zeta' \equiv i|\zeta'|e^{i\phi}.$$

Since $\zeta \in \mathscr{Z}$, we can choose ϕ to be an odd function whose range lies in $(-\pi/2, \pi/2)$. It is easy to show that if $\zeta \in C^{1,\alpha}$, then $\phi \in C^{0,\alpha}$. Let $\lambda[\zeta](s)$ be the arc length from $\zeta(0)$ to $\zeta(s)$:

$$\text{(3.16)} \qquad \lambda[\zeta](s) \equiv \int_0^s |\zeta'(t)|\, dt.$$

In the sequel we shall need the following two lemmas, the first of which collects well-known facts on the Schwarz-Poisson kernel and on the Reflection Principle (cf. Rudin (1966, Theorems 11.8, 11.11, 11.12, and 11.17), Pommerenke (1975, Lemma 10.2), and Birkhoff and Zarantonello (1957, p. 136)).

3.17. **Lemma.** *Let* $u \in C^{0,\alpha}[-\pi/2, \pi/2]$, $\alpha \in (0, 1]$, *and*

$$\text{(3.18)} \qquad r[u](\omega) \equiv \frac{1+\omega^2}{\pi} \int_{-\pi/2}^{\pi/2} \frac{u(\tau)}{1 + 2i\omega \sin\tau - \omega^2}\, d\tau, \qquad |\omega| < 1.$$

The function $r[u](\cdot)$ *can be extended by continuity to* $\operatorname{cl}\mathscr{D}(0, 1)$. *Such an extension is denoted by the same symbol* $r[u](\cdot)$. *If* $\alpha \in (0, 1)$, *then*

$r[u] \in C^{0,\alpha}(\mathrm{cl}\mathscr{D}(0, 1)) \cap \mathrm{H}(\mathscr{D}(0, 1))$. *If* $\alpha = 1$, *then* $r[u] \in C^{0,\beta}(\mathrm{cl}\mathscr{D}(0, 1)) \cap \mathrm{H}(\mathscr{D}(0, 1))$ $\forall \beta \in (0, 1)$. *Furthermore,*

$$\text{(3.19a)} \qquad \mathrm{Re}\, r[u](e^{it}) = u(t);$$

$$\text{(3.19b)} \qquad \mathrm{Im}\, r[u](e^{it}) = -\frac{\cos t}{\pi} \int_{-\pi/2}^{\pi/2} \frac{u(t) - u(\tau)}{\sin t - \sin \tau}\, d\tau \quad \text{if } t \in [-\pi/2, \pi/2];$$

$$\text{(3.19c)} \qquad \text{if } |\omega| < 1 \text{ and if } \mathrm{Re}\,\omega = 0, \text{ then } \mathrm{Im}\, r[u](\omega) = 0;$$

$$\text{(3.19d)} \qquad r[u](\omega) = \overline{r[u](-\overline{\omega})} \quad \text{if } \omega \in \mathrm{cl}\mathscr{D}(0, 1).$$

The following lemma was derived by Osgood (1928, p. 400) from a well-known theorem attributed to Darboux, which we state and comment on in §4.

3.20. **Lemma.** *Let* $\Gamma \in C^0([0, 1], \mathbb{C})$ *be a simple closed curve and* $\mathscr{I}[\Gamma]$ *be the bounded connected component of* $\mathbb{C} \setminus \Gamma([0, 1])$. *Let* $\omega_0 \in \partial\mathscr{I}[\Gamma]$, $f \in C^0(\mathrm{cl}\mathscr{I}[\Gamma] \setminus \{\omega_0\}) \cap \mathrm{H}(\mathscr{I}[\Gamma])$, *and* $\lim_{z \to \omega_0} f = \infty$. *If there exists at least one point* $z_0 \in \mathbb{C} \setminus f(\mathrm{cl}\mathscr{I}[\Gamma] \setminus \{\omega_0\})$ *and if* f *is injective on* $\partial\mathscr{I}[\Gamma] \setminus \{\omega_0\}$, *then* f *is a homeomorphism of* $\mathrm{cl}\mathscr{I}[\Gamma] \setminus \{\omega_0\}$ *onto* $\mathrm{cl}\,\Sigma$, *where* Σ *denotes the (open) connected component of* $\mathbb{C} \setminus f(\partial\mathscr{I}[\Gamma] \setminus \{\omega_0\})$ *that does not contain* z_0. *The same result holds if* $\mathscr{I}[\Gamma]$ *is replaced by an unbounded open set* $\mathscr{A}$ *and if* ω_0 *is replaced by* ∞, *provided that there exists a biholomorphic map* g *of* $\mathscr{A}$ *onto a set* $\mathscr{I}[\Gamma]$, *as in the first part of the statement, that can be extended by continuity to* $\mathrm{cl}\mathscr{A}$.

We can now prove

3.21. **Theorem.** *Let* $U > 0$, $\alpha \in (0, 1]$, *and* $\zeta \in C^{1,\alpha} \cap \mathscr{Z}$. *If* f *is a solution of CMP, then the function* l *in* $C^1[-\pi/2, \pi/2]$ *defined by*

$$\text{(3.22)} \qquad f(e^{it}) = \zeta(\lambda[\zeta]^{-1}(l(t)))$$

satisfies Villat's equation

$$\text{(3.23a)} \qquad l(t) = b[\zeta, l] \int_0^t e^{\chi[\zeta, l](\tau)}(1 + \cos\tau)\cos\tau\, d\tau,$$

where

$$\text{(3.23b)} \qquad b[\zeta, l] \equiv \frac{\lambda[\zeta](L)}{\int_0^{\pi/2} e^{\chi[\zeta, l](\tau)}(1 + \cos\tau)\cos\tau\, d\tau},$$

$$\text{(3.23c)} \qquad \chi[\zeta, l](\tau) \equiv -\mathrm{Im}\, r[\phi[\zeta](\lambda^{-1}[\zeta](l))](e^{i\tau}),$$

and $r[\cdot]$ *is the linear operator defined in Lemma* 3.17. *Conversely, if an odd function* l *in* $C^1[-\pi/2, \pi/2]$ *satisfying Villat's equation is given, then there exists a unique function* $f \in C^0(\mathrm{cl}\mathscr{H} \setminus \{0\})$, *satisfying CMP and* (3.22). *It is defined by*

$$\text{(3.24a)} \qquad f'(\omega) = \frac{b[\zeta, l](1 + \omega)^2(1 + \omega^2)}{4\omega^3} \exp\{i r[\phi[\zeta](\lambda^{-1}[\zeta](l))](\omega)\}, \qquad \omega \in \mathscr{H},$$

$$\text{(3.24b)} \qquad f(\pm i) = \pm i.$$

Furthermore, if CMP were to have distinct solutions f_1 and f_2, the corresponding l_1 and l_2 would be different.

Proof. Let CMP have a solution f and let the function N be defined as

$$(3.25)\qquad N(\omega) \equiv UM^{-1}\omega^3 f'(\omega)(1+\omega)^{-2}(1+\omega^2)^{-1} \qquad \omega \in \mathrm{cl}\,\mathscr{H}\backslash\{0, \pm i\}.$$

By virtue of (3.5a, b, d, e, j), the holomorphic function N can be extended by continuity to cl $\mathscr{H}$ and this extension has no zeros. Since $\mathscr{H}$ is open and simply-connected, and since $N(\omega) \neq 0$ if $\omega \in \mathscr{H}$, there exists (cf. Ahlfors (1973, p. 135)) an $r \in \mathrm{H}(\mathscr{H})$ (defined up to an integer multiple of 2π) such that

$$(3.26)\qquad e^{ir(\omega)} = N(\omega), \qquad \omega \in \mathscr{H}.$$

We now show that (any such) r can be extended by continuity to cl $\mathscr{H}$. Let $\omega_1 \in \partial\mathscr{H}$. Since $N(\omega_1) \neq 0$ and N is continuous at ω_1, there exists an $\varepsilon > 0$ and a closed disk $\mathrm{cl}\,\mathscr{D}(\omega_1, \varepsilon)$ such that a branch of log can be defined in a neighborhood of $N(\mathrm{cl}\,\mathscr{D}(\omega_1, \varepsilon) \cap \mathrm{cl}\,\mathscr{H})$. By the connectedness of $\mathscr{D}(\omega_1, \varepsilon) \cap \mathscr{H}$, we have

$$(3.27)\qquad ir(\omega) - \log N(\omega) = 2\pi m i, \qquad \omega \in \mathscr{D}(\omega_1, \varepsilon) \cap \mathscr{H},$$

for some integer m. From (3.27) it follows that r can be extended by continuity to $\mathrm{cl}\,\{\mathscr{D}(\omega_1, \varepsilon) \cap \mathscr{H}\}$. By the arbitrariness of ω_1, the function r can be extended by continuity to cl $\mathscr{H}$. By (3.22), (3.25), and (3.5i,j,k), we have

$$(3.28\mathrm{a})\qquad N(e^{it}) = \frac{-iU}{8M\cos t\cos^2(t/2)}\zeta'(k(t))k'(t), \qquad t \in (-\pi/2, \pi/2),$$

where

$$(3.28\mathrm{b})\qquad k(t) \equiv \lambda^{-1}[\zeta](l(t)),$$

$$(3.29)\qquad |N(it)| = 1, \qquad t \in [-1, 1].$$

Since f is a homeomorphism of $\{e^{it} : t \in (-\pi/2, \pi/2)\}$ onto $\zeta(-L, L)$, since $f'(e^{it}) \neq 0$ if $t \in (-\pi/2, \pi/2)$, and since $f(\pm i) = \pm i$, it follows that we have $k'(t) > 0$ if $t \in (-\pi/2, \pi/2)$. Hence from (3.15), (3.26), and (3.28) we deduce that

$$(3.30)\qquad \mathrm{Re}\, r(e^{it}) - \phi(k(t)) = 2\pi n \quad \forall t \in (-\pi/2, \pi/2)$$

for some integer n. We choose r in (3.26), (3.30) so that $n = 0$. By (3.26), (3.29), we have

$$(3.31)\qquad \mathrm{Im}\, r(it) = 0 \quad \forall t \in [-1, 1].$$

Hence, we can use the Schwarz Reflection Principle to continue r analytically onto the unit disk $\mathscr{D}(0, 1)$. We denote this extension also by r. Since $r \in C^0(\mathrm{cl}\,\mathscr{H}) \cap \mathrm{H}(\mathscr{H})$, the extended r belongs to $C^0(\mathrm{cl}\,\mathscr{D}(0, 1)) \cap \mathrm{H}(\mathscr{D}(0, 1))$. By the Identity Principle and Lemma 3.17, we have

$$(3.32)\qquad r[\phi(k)](\omega) = r(\omega), \qquad \omega \in \mathrm{cl}\,\mathscr{D}(0, 1).$$

Hence, (3.25), (3.26), and (3.32) imply that (3.24a) is satisfied with $b[\zeta, l]$ replaced by $4MU^{-1}$. By letting $\omega \to e^{it}$, we obtain

$$(3.33)\quad f'(e^{it})e^{it} = \frac{4M}{U}\cos t(1+\cos t)\exp\{ir[\phi(k)](e^{it})\}, \qquad t \in (-\pi/2, \pi/2).$$

Since $l'(t) = |f'(e^{it})|$, by taking the modulus in (3.33), and integrating the resulting equation subject to the boundary conditions $l(0) = 0$, $l(\pi/2) = \lambda[\zeta](L)$ (derived from (3.22), (3.5g)), we evaluate M in terms of l and thereby obtain Villat's equation.

Let $f_1 \neq f_2$ be two distinct solutions of CMP. Since the equality of the corresponding l_1 and l_2 defined as in (3.22) together with (3.24) would imply that $f_1 = f_2$, we must necessarily have $l_1 \neq l_2$.

Conversely, assume that l is given. By the membership of $\phi \in C^{0,\alpha}$, by Lemma 3.17, and by (3.24) it follows that conditions (3.5d,e,i) hold and that $f \in C^0(\mathrm{cl}\mathscr{H}\backslash\{0\}) \cap \mathrm{H}(\mathscr{H})$. Conditions (3.5j,k) can be verified by direct computation. By letting $\omega = e^{it}$ in (3.24a) and using Lemma 3.17 together with Villat's equation, we obtain

$$(3.34)\qquad e^{it}f'(e^{it}) = l'(t)e^{i\phi(k(t))}, \qquad t \in (-\pi/2, \pi/2).$$

Villat's equation implies that $l(\pm\pi/2) = \pm\lambda[\zeta](L)$, $l' > 0$. By using (3.15) and integrating (3.34) from $t = -\pi/2$ to t subject to $f(-i) = -i$, $l(-\pi/2) = -\lambda[\zeta](L)$, we obtain (3.22) and consequently (3.5h).

We now prove the second part of (3.5g). From (3.24a) and the membership of $r[\phi(k)]$ in $C^{0,\beta}(\mathrm{cl}\mathscr{D}(0,1)) \cap \mathrm{H}(\mathscr{D}(0,1))$, $\beta \in (0, \alpha)$, we deduce that there exists a unique function $\rho \in C^{0,\beta}(\mathrm{cl}\mathscr{D}(0,1)) \cap \mathrm{H}(\mathscr{D}(0,1))$ such that

$$(3.35\text{a,b})\qquad f'(\omega) = \frac{\rho(\omega)}{\omega^3}, \qquad \rho(0) = \frac{M}{U}\ (\neq 0).$$

Then there exists a unique $v \in C^{0,\beta}(\mathrm{cl}\mathscr{D}(0,1)) \cap \mathrm{H}(\mathscr{D}(0,1))$ satisfying

$$(3.36)\qquad (v(\omega)\omega^{-2})' = f'(\omega) - \frac{\rho''(0)}{2\omega}, \qquad v''(0) = 0.$$

Then

$$(3.37)\qquad f(\omega) = c + \frac{v(\omega)}{\omega^2} + \frac{\rho''(0)}{2}\log\omega, \qquad \omega \in \mathscr{D}(0,1)\backslash(-\infty, 0],$$

where the constant c can be determined by the condition $f(\pm i) = \pm i$ and where log is the principal branch of log (so that $\mathrm{Im}\log\omega \in (-\pi, \pi)$). By differentiating (3.37) and using (3.35) we find that $v(0) = -M/(2U)$. Then (3.37) yields

$$(3.38)\qquad \lim_{\omega\to 0}\omega^2 f(\omega) = -\frac{M}{2U} \neq 0.$$

Hence $\lim_{\omega\to 0} f(\omega) = \infty$ and the proof of (3.5g) is complete.

We now prove (3.5f). Clearly

$$\tau^\sharp(t) = \int_{-1}^{t} \frac{|f'(-i\eta)|}{|f^2(-i\eta)|}\,d\eta. \tag{3.39}$$

From (3.24a) and (3.37), we readily deduce that

$$\lim_{t\to 0^-} t^2 f(-it) \in \mathbb{C}\backslash\{0\}, \qquad \lim_{t\to 0^-} t^3 f'(-it) = \frac{M}{Ui} \in \mathbb{C}\backslash\{0\}. \tag{3.40a, b}$$

Then (3.39) implies that $\tau^\sharp(0) < \infty$. We now define $\mu(t) \equiv 1/f(-it)$. By the standard properties of the composition of Hölder continuous functions, (3.5f) can be deduced from

$$\frac{\mu'}{|\mu'|} \in C^{0,\beta}[-1, 0] \quad \forall \beta \in (0, \alpha), \qquad t^\sharp \in C^{0,1/2}[0, \tau^\sharp(0)]. \tag{3.41a, b}$$

We first consider (3.41a). By simple computations we deduce from (3.24a) that

(3.42)
$$\frac{\mu'}{|\mu'|}(t) = \frac{1-it}{1+it}|t^4 f^2(-it)| t^{-4} f^{-2}(-it) \exp ir[\phi(k)](-it), \qquad t \in [-1, 0).$$

Condition (3.40a) implies that $\mu'/|\mu|$ can be extended by continuity to $[-1, 0]$ and that such an extension does not vanish. Furthermore, the inclusions $\phi \in C^{0,\alpha}$, $\lambda^{-1} \in C^1$, and $l \in C^1$, together with Lemma 3.17 imply that the function $t \mapsto r[\phi(k)](-it)$ belongs to $C^{0,\beta}[-1, 0]$ $\forall \beta \in (0, \alpha)$. Hence the Hölder continuity of $\mu'/|\mu'|$ follows from that of $f(-it)t^2$, which we can deduce from (3.37) and the membership of $\upsilon \in C^{0,\beta}(\mathrm{cl}\mathscr{D}(0, 1))$, $\beta \in (0, \alpha)$. Note that (3.39), (3.24) imply that

$$\begin{aligned} \lim_{t\to -1} \frac{|\tau^\sharp(t)|^{1/2}}{|\frac{d\tau^\sharp}{dt}(t)|} &= \lim_{t\to -1} \left|\frac{t+1}{\frac{d\tau^\sharp}{dt}(t)}\right| \left|\frac{\tau^\sharp(t)}{(t+1)^2}\right|^{1/2} \\ &= \frac{1}{\sqrt{2}} \lim_{t\to -1} \left|\frac{t+1}{\frac{d\tau^\sharp}{dt}(t)}\right|^{1/2} \in \mathbb{C}\backslash\{0\}. \end{aligned} \tag{3.43}$$

Hence Lemma 2.13 implies that $t^\sharp \in C^{0,1/2}[0, \tau^\sharp(0)/2]$. Similarly, we can show that $t^\sharp \in C^{0,1/2}[\tau^\sharp(0)/2, \tau^\sharp(0)]$ by noting that

$$\lim_{t\to 0^-} |\tau^\sharp(t) - \tau^\sharp(0)|^{1/2} \left|\frac{d\tau^\sharp}{dt}(t)\right|^{-1} \in \mathbb{C}. \tag{3.44}$$

We now prove (3.5b). As a first step, we show that

$$\mathrm{Im}\{-if'(-it)\} > 0 \quad \text{if } t \in (-1, 0). \tag{3.45}$$

Our proof of (3.45) is based on a combination of ideas of Serrin (1952a) and Leray (1935–36). By setting $\omega = -it$, $t \in (-1, 0)$, in (3.24a), we easily obtain that

$$\frac{d}{dt}\{f(-it)\} = |f'(-it)| \exp\{i\beta(t)\}, \qquad t \in (-1, 0), \tag{3.46a}$$

where

$$\beta(t) \equiv \arctan \frac{2t}{t^2 - 1} + \operatorname{Re} r[\phi(k)](-it). \tag{3.46b}$$

Clearly $\arctan \frac{2t}{t^2-1} \in (0, \pi/2)$ if $t \in (-1, 0)$. Since $\zeta \in \mathscr{Z} \cap C^{1,\alpha}$, it follows that $\phi \in C^{0,\alpha}$ and $\operatorname{Re} r[\phi(k)](e^{it}) = \phi(k)(t) \in (-\pi/2, \pi/2)$ if $t \in [-\pi/2, \pi/2]$. Then by the Maximum Principle applied to the harmonic function $\operatorname{Re} r[\phi(k)](\omega)$ in $\operatorname{cl} \mathscr{D}(0, 1)$, we have

$$\sup_{t\in(-1,0)} |\operatorname{Re} r[\phi(k)](-it)| \leq \sup_{|\omega|=1} |\operatorname{Re} r[\phi(k)](\omega)| < \frac{\pi}{2}. \tag{3.47}$$

Thus $\beta(t) \in (-\pi/2, \pi)$. Let $\tilde{\sigma}(\tau) = \pi/2$ if $\tau \in (0, \pi/2]$, and $\tilde{\sigma}(\tau) = -\pi/2$ if $\tau \in [-\pi/2, 0]$. By standard properties of the Schwarz-Poisson kernel, we have $r[\tilde{\sigma}] \in \mathscr{H}(\mathscr{D}(0, 1))$, and $\lim_{x\to 1^-} r[\tilde{\sigma}](xe^{it}) = \tilde{\sigma}(t)$ if $t \in (-\pi, 0) \cup (0, \pi)$. Hence $ir[\tilde{\sigma}](\omega) = \log((1+\omega)/(1-\omega))$, $|\omega| < 1$ (with $\operatorname{Im} \log \omega \in (-\pi, \pi)$). Then (3.24a) implies that the function β can be written as

$$\beta(t) = \frac{(1-t^2)}{\pi} \operatorname{Re} \int_{-\pi/2}^{\pi/2} \psi(\tau) \left\{ \frac{1}{1 + 2t \sin \tau + t^2} - \frac{1}{1+t^2} \right\} d\tau, \tag{3.48a}$$

where

$$\psi(\tau) \equiv \phi(k)(\tau) + \tilde{\sigma}(\tau). \tag{3.48b}$$

Hence, the positivity of β follows because ψ and the term in braces in (3.48a) each have the same sign as $\tau \in [-\pi/2, \pi/2] \setminus \{0\}$ and cannot vanish identically. Thus $\beta(t) \in (0, \pi)$. Since $if'(-it) \neq 0$ $\forall t \in (-1, 0)$, we conclude that (3.45) holds. From (3.45), the membership of $\zeta \in \mathscr{Z}$, and (3.5i) we easily deduce that

$$\operatorname{Im} f(\omega) > 0 \text{ if } \omega \in \mathscr{J}^+ \cap \partial\mathscr{H}, \tag{3.49a}$$

$$f(\partial\mathscr{H} \setminus \{0\}) \cap \mathbb{R} = \{\zeta(0)\}, \tag{3.49b}$$

$$f \text{ is injective on } \partial\mathscr{H} \setminus \{0\}. \tag{3.49c}$$

By virtue of Lemma 3.20, the injectivity of f on $\partial\mathscr{H} \setminus \{0\}$ implies (3.5b) provided that we can prove the existence of at least one point in $\mathbb{C} \setminus f(\operatorname{cl} \mathscr{H} \setminus \{0\})$. In fact, we prove that

$$(\zeta(0), \infty) \subseteq \mathbb{C} \setminus f(\operatorname{cl} \mathscr{H} \setminus \{0\}). \tag{3.50}$$

From (3.5g), (3.22), and (3.24a), we deduce that $\zeta(0) = f(1)$, $\lim_{t\to 0^+} |f(t)| = \infty$, and $f'((0, 1)) \subseteq (0, \infty)$. Hence

$$f((0, 1]) = (-\infty, \zeta(0)]. \tag{3.51}$$

By virtue of (3.49b) and (3.51), the condition

$$\operatorname{Im} f(\omega) \neq 0 \quad \text{if } \omega \in \mathscr{H}, \qquad \operatorname{Im} \omega \neq 0, \tag{3.52}$$

implies (3.50). We now prove (3.52). Let h be the function defined in (3.2) and let $\kappa \equiv f \circ h^{-1}$. Condition (3.5i) and the definition of κ imply that

(3.53) $$\operatorname{Im} \kappa(\eta) \neq 0, \qquad \eta \in \mathscr{J}^+,$$

is equivalent to (3.52). We can deduce the validity of (3.53) from the following two conditions, which we prove below:

(3.52a) $$\frac{\partial}{\partial x}\{\operatorname{Im} \kappa(x+iy)\} > 0, \qquad (x, y) \in \mathbb{R} \times (0, \infty),$$

(3.54b) $$\lim_{(x,y)\to\infty} \frac{\partial}{\partial y}\{\operatorname{Im} \kappa(x+iy)\} > 0.$$

To show that (3.54) implies (3.53), let $\overline{\eta} \in \mathscr{J}^+$. By (3.54b) there exists a real number $x_1 < \min\{\operatorname{Re} \overline{\eta}, 0\}$ such that $\frac{\partial}{\partial y}\operatorname{Im} \kappa(\eta) > 0$ for $\eta \in [x_1, x_1 + i\operatorname{Im} \overline{\eta}]$. Since $\operatorname{Im} \kappa(\eta) = 0$ when $\eta \in (-\infty, 0]$, it follows that $\operatorname{Im} \kappa(x_1 + i\operatorname{Im} \overline{\eta}) > 0$. Hence (3.54a) implies that $\operatorname{Im} \kappa(\overline{\eta}) \neq 0$. We now prove (3.54). Let

(3.55) $$\nu(\eta) \equiv \left\{\frac{d}{d\eta}\kappa(\eta)\right\}^{-1}.$$

From (3.2), (3.24a), the membership of $\phi(k)$ in $C^{0,\alpha}$, and Lemma 3.17, we easily find that

(3.56a) $$\nu(\eta) = \frac{U}{2M}\frac{1-h^{-1}(\eta)}{1+h^{-1}(\eta)}\exp\{-ir[\phi(k)](h^{-1}(\eta))\},$$

(3.56b) $$\nu \in C^0(\operatorname{cl}\mathscr{J}^+) \cap \mathrm{H}(\mathscr{J}^+),$$

(3.56c) $$\lim_{\eta\to\infty} \nu(\eta) = \frac{U}{2M} \in (0, \infty),$$

(3.56d) $$\operatorname{Im} \nu(\eta) = 0 \quad \text{if } \eta \in (-\infty, 0].$$

From (3.56c) we immediately deduce (3.54b). Since every $\eta \in (0, 1]$ can be written in the form $\eta = h(e^{it})$, $t \in (0, \pi/2]$, and every $\eta \in [1, \infty)$ can be written in the form $\eta = h(-it)$, $t \in (-1, 0)$, conditions (3.19a), (3.45), and (3.56a) imply

(3.57a)
$$\operatorname{Im} \nu(\eta) = \frac{-U}{2M}\cos\phi(k(t))\tan\frac{t}{2}\exp\{\operatorname{Im} r[\phi(k)](e^{it})\} < 0 \quad \text{if } \eta \in (0, 1],$$

(3.57b) $$\operatorname{Im} \nu(\eta) = \operatorname{Im}\left\{\frac{t^4-1}{t^3(-if'(-it))}\right\} < 0, \qquad \eta \in [1, \infty).$$

By applying the Maximum Principle to the function $\operatorname{Im} \nu$ on $\operatorname{cl}\mathscr{D}(0, R) \cap \operatorname{cl}\mathscr{J}^+$, $R > 0$, we can show by contradiction that (3.56b,c,d) and (3.57) imply

(3.58) $$\operatorname{Im} \nu(\eta) \le 0 \quad \text{if } \operatorname{Im} \eta > 0.$$

Then, by reapplying the Maximum Principle on $\operatorname{cl}\mathscr{D}(0, R) \cap \operatorname{cl}\mathscr{J}^+$ we conclude that

(3.59) $$\operatorname{Im} \nu(\eta) < 0 \quad \text{if } \operatorname{Im} \eta > 0,$$

which in turn implies (3.54a) (cf. (3.55)). □

4. A comment on the Darboux theorem

The following well-known result can be derived from the Argument Principle. Osgood (1928, p. 397) attributes it to G. Darboux (1887, p. 173), whose presentation is not compelling to a modern reader. In this work Darboux gives no indication that the result is new and due to him.

4.1. **Theorem.** *Let* $\Gamma \in C^0([0, 1], \mathbb{C})$ *be a simple closed curve and let* $\mathscr{I}[\Gamma]$ *be the bounded connected component of* $\mathbb{C}\backslash\Gamma([0, 1])$. *Let* $f \in C^0(\mathrm{cl}\,\mathscr{I}[\Gamma]) \cap \mathrm{H}(\mathscr{I}[\Gamma])$. *If* f *is injective on* $\partial\mathscr{I}[\Gamma]$, *then* f *is a homeomorphism of* $\mathrm{cl}\,\mathscr{I}[\Gamma]$ *onto its image in* $\mathbb{C}$.

With the following examples, we show that the same result does not hold either if the domain of f is unbounded or if f tends to ∞ at a point on the boundary of the domain of f unless more restrictive hypotheses are made on the function f, as done by Osgood (cf. Lemma 3.20.) We then discuss their implications for cavitation.

4.2. **Counterexamples.**

(4.2a) Let $\Gamma^*(t) \equiv it$, $t \in \mathbb{R}$, let $\mathscr{I}^*[\Gamma]$ be the (unbounded) domain $\{z \in \mathbb{C}: \mathrm{Re}\, z < 0\}$, and let $f_1(z) \equiv z^5$. Clearly $f_1 \in C^0(\mathrm{cl}\,\mathscr{I}^*[\Gamma]) \cap \mathrm{H}(\mathscr{I}^*[\Gamma])$, f_1 is a homeomorphism of $\partial\mathscr{I}^*[\Gamma]$ onto itself, $\lim_{z\to\infty} z^5 = \infty$, and $\mathscr{I}^*[\Gamma]$ satisfies the second part of the statement of Lemma 3.20. Nevertheless $f_1(\mathscr{I}^*[\Gamma]) = \mathbb{C}$ and f_1 is not injective.

(4.2b) Let $\Gamma \in C^0([0, 1])$ parametrize $\partial\mathscr{H}$, $\mathscr{I}[\Gamma] = \mathscr{H}$, and $f_2(\omega) \equiv [(\omega^2 - 1)/2\omega]^5$. Clearly $f_2 \in C^0(\mathrm{cl}\,\mathscr{I}[\Gamma]\backslash\{0\}) \cap \mathrm{H}(\mathscr{I}[\Gamma])$, and $\lim_{\omega\to 0} f_2(\omega) = \infty$. Finally, since $(\omega^2 - 1)/2\omega$ is a homeomorphism of $\mathrm{cl}\,\mathscr{H}\backslash\{0\}$ onto $\mathrm{cl}\,\mathscr{I}^*[\Gamma]$ and maps $\partial\mathscr{H}\backslash\{0\}$ onto $i\mathbb{R}$, the previous example shows that f_2 is injective on $\partial\mathscr{H}\backslash\{0\}$. Nevertheless $f_2(\mathscr{I}[\Gamma]) = \mathbb{C}$ and f_2 is not injective.

We now turn to Villat's equation. It seems to be generally accepted in the literature on cavitational flows, that once a solution l of Villat's equation is given, then by integrating (3.24) a map f can be obtained which in turn produces by (3.6) a solution of the Flow Problem. Counterexample (4.2b) shows that the injectivity of f on $\partial\mathscr{H}\backslash\{0\}$ (which physically means that the curve represented by the obstacle ζ and the streamlines γ, $\overline{\gamma}$ does not intersect itself) does not guarantee by itself that f be injective. For this reason we had to prove global injectivity of f in the proof of Theorem 3.21.

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